'Getting Started' Questions for STEP Section 10 - Mechanics

1 A light aluminium ladder AB of length $6a$ rests with one end A in contact with rough horizontal ground; a point C of the ladder is in contact with the top of a vertical wall of height $4a$ (so that the ladder projects over the wall) and the foot A of the ladder being at a distance $3a$ from the wall. The vertical plane containing the ladder is perpendicular to the wall. A man of weight W stands on the ladder at a distance x from A, the system being in equilibrium. Assuming that the contact C is smooth, show (by taking moments about A, or otherwise) that the force exerted by the wall on the ladder is $\dfrac{3Wx}{25a}$.

Find also R, the normal component of reaction, and F, the frictional force, at A in terms of W, a and x.

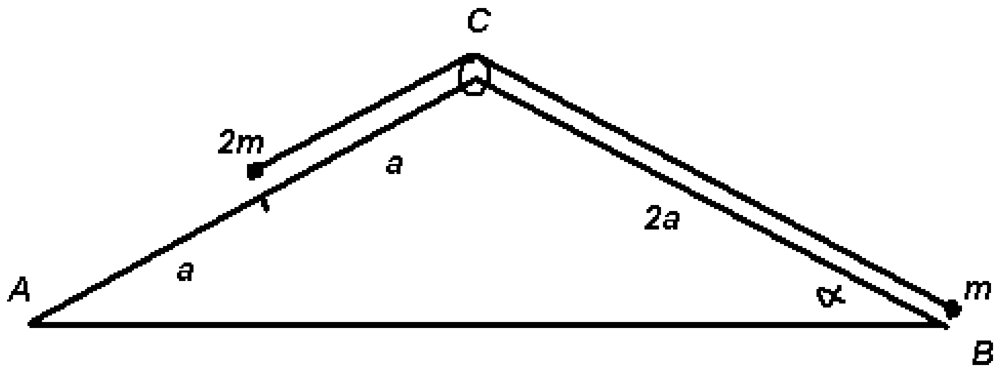
The man can reach C without the ladder's slipping. Find the least possible value for the coefficient of friction at A.

2 Three particles A, B and C of masses m, $2m$ and $3m$ respectively, lie at rest in a straight line in that order on a smooth horizontal table, the coefficient of restitution between each pair of particles being e. The particle A is projected directly towards B with speed u, and immediately after the collision with B the speed of B is $\dfrac{7}{12}u$. Calculate e.

As soon as the collision takes place, C is projected with speed $\dfrac{1}{8}u$ towards B. Show that just after B and C collide, the speed of B is $\dfrac{77}{480}u$, and find the speed of C.

Show that there are no more collisions.

3



The diagram shows the cross-section *ABC* of a wedge resting on rough horizontal ground; the plane *ABC* is vertical. The faces, *AC*, *BC* of the wedge are also rough and each is inclined at an angle α to the ground. $AC = BC = 2a$. A light inextensible string of length $3a$ passes over a small smooth pulley at *C* and has particles of masses $2m$ and m attached to its ends. Initially the particle of mass $2m$ is distant a from *C* and the string is taut. The coefficient of friction between each particle and the faces of the wedge is μ. The system is released from rest and it is observed that the particles move *but the wedge remains stationary*. Find:

(i) the acceleration of the particles,
(ii) the tension in the string,
(iii) the frictional force exerted by the ground on the wedge.

Show that, for motion to occur, $\mu < \frac{1}{3}\tan\alpha$.

If $\mu = \frac{1}{4}\tan\alpha$, find the speed of the particle of mass $2m$ when it arrives at *A*. At the instant of arrival at *A* the string breaks. Show that the particle of mass *m* ascends a further distance $\frac{1}{15}a$ up *BC* and find the speed when it returns to its starting point.

4 A particle A of mass m is attached to one end of an elastic string of natural length a and modulus λ. The other end of the string is attached to a point O on a smooth horizontal table. Initially A is at rest on the table at a point P which is at a distance a from O. A second particle B, also of mass m, is projected with speed u along the table in the direction PO so as to strike A.

Show that, just after the impact, A has speed $\frac{1}{2}(1+e)u$ and B has speed $\frac{1}{2}(1-e)u$, where e is the coefficient of restitution between the particles.

It is observed that the second impact between the particles occurs at the point O. Prove that, after the second impact, A is at rest and B has speed eu in the direction OP.

Find u and deduce that $e > \frac{1}{2}$.

5 A particle of mass m is held in contact with the smooth inclined face of a wedge of mass M; the wedge is at rest on a rough horizontal table. The inclined face of the wedge is at an angle α to the horizontal, and the coefficient of friction between the wedge and the table is μ. When the particle is released from rest, the wedge commences to move along the table.

(i) Prove that $\mu < \dfrac{R\sin\alpha}{Mg + R\cos\alpha}$ and that $R < mg\cos\alpha$, where R is the reaction between the wedge and the particle. Hence deduce that

$$\mu < \frac{m\cos\alpha\sin\alpha}{M + m\cos^2\alpha}.$$

(ii) Prove that the acceleration of the particle relative to the wedge is

$$\frac{g(M+m)(\sin\alpha - \mu\cos\alpha)}{(M + m\sin^2\alpha - \mu m\sin\alpha\cos\alpha)}$$

down the inclined face.

6 The upper end A of a uniform rod AB of mass m and length $2a$ is freely pivoted to a fixed point. A bead C of mass $\frac{1}{12}m$ slides on a smooth horizontal wire passing through A, the bead being connected to B by a light inextensible string of length $2a$. In a general position the rod makes an angle θ with the downward vertical at A, the string is taut, and B is vertically below the mid-point of AC. Prove that, if the system is in motion, then

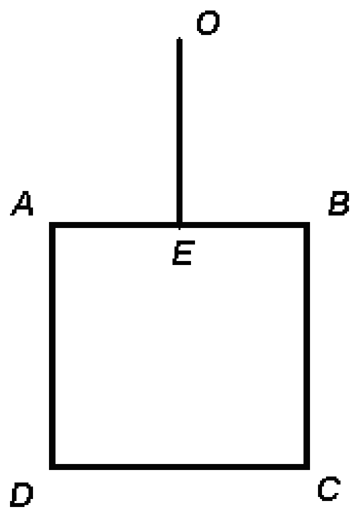
$$2a\left(\frac{d\theta}{dt}\right)^2(1+\cos^2\theta) = 3g\cos\theta + constant\ .$$

(i) Initially the rod is vertical and the system if in motion, the bead having speed $(6ag)^{\frac{1}{2}}$ along the wire. Find the angle turned through by the rod before it first comes to instantaneous rest.
(ii) Initially the rod is vertical and the bead is slightly displaced from its equilibrium position and released from rest. Show that the resulting motion is approximately simple harmonic, having period $4\pi\sqrt{\frac{2a}{3g}}$.

7 The polar coordinates of a particle moving in a plane are (r,θ) at time t. You may assume that the radial and transverse components of acceleration are $\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$ and $\frac{1}{r}\frac{d}{dt}\left(r^2\left(\frac{d\theta}{dt}\right)\right)$.

If $\frac{d\theta}{dt}$ is constant and equal to ω, show that the transverse expression reduces to $2\omega\frac{dr}{dt}$, and write down the reduced form of the radial expression.
A straight smooth tube of small bore contains a particle of mass m which is attached to a fixed point O inside the tube by means of an elastic string of natural length a and modulus $2ma\omega^2$. The tube is forced to rotate about O with contant angular velocity ω so that the tube moves in a horizontal plane through O. Initially the string is just taut and the particle is at rest relative to the tube. Show that, in the subsequent motion, the particle oscillates along a length $2a$ of the tube, and describe with the help of a rough sketch the path of the particle relative to the plane. Show also that the couple required to maintain the motion at time t from the start of the motion is $ma\omega^2(4\sin\omega t - \sin 2\omega t)$.

8



The diagram shows a rigid body formed from a uniform rod *OE* of mass *m* and length 4*a*, and a uniform square lamina of mass 4*m* and side 6*a*. The end *E* of the rod is the mid-point of *AB;* the rod is perpendicular to *AB* and in the plane of the lamina. The body hangs in equilibrium and the end *O* of the rod is smoothly pivoted to a fixed point. Find the moment of inertia of the body about an axis though *O* perpendicular to the plane of the lamina, and the distance of the centre of gravity of the body from *O*.
The body is set in motion by being given an initial angular velocity Ω about the axis so that *B* starts to rise. Show that, when the body has swung though an angle θ,

$$\left(\frac{d\theta}{dt}\right)^2 = \Omega^2 - \frac{45g}{169a}(1-\cos\theta).$$

Hence show that, if $18 < \frac{169a\Omega^2}{g} < 90$, *B* will rise above the level of *O*, but the body will not make complete revolutions.
If $\Omega^2 = \frac{72g}{169a}$, find the angle through which the body swings before first coming to instantaneous rest, and its angular acceleration when in this position of instantaneous rest.